

Transfinite diameter, Chebyshev constant and energy on locally compact spaces

B\'alint Farkas* (farkas@mathematik.tu-darmstadt.de)
*Technische Universit\"at Darmstadt, Fachbereich Mathematik
 Department of Applied Analysis
 Schlo\ßgartenstra\se 7, D-64289, Darmstadt, Germany*

B\'ela Nagy† (nbela@sol.cc.u-szeged.hu)
*Bolyai Institute, University of Szeged
 Aradi v\'ertan\'uk tere 1
 H-6720, Szeged, Hungary*

Abstract. We study the relationship between transfinite diameter, Chebyshev constant and Wiener energy in the abstract linear potential analytic setting pioneered by Choquet, Fuglede and Ohtsuka. It turns out that, whenever the potential theoretic kernel has the maximum principle, then all these quantities are equal for all compact sets. For continuous kernels even the converse statement is true: if the Chebyshev constant of any compact set coincides with its transfinite diameter, the kernel must satisfy the maximum principle. An abundance of examples is provided to show the sharpness of the results.

Keywords: Transfinite diameter, Chebyshev constant, energy, potential theoretic kernel function in the sense of Fuglede, Frostman's maximum principle, rendezvous and average distance numbers.

2000 Math. Subj. Class.: 31C15; 28A12, 54D45

Dedicated to the memory of Professor Gustave Choquet (1 March 1915 - 14 November 2006)

1. Introduction

The idea behind abstract (linear) potential theory, as developed by Choquet [4], Fuglede [9] and Ohtsuka [15], is to replace the Euclidian space \mathbb{R}^d by some locally compact space X and the well-known Newtonian kernel by some other kernel function $k : X \times X \rightarrow \mathbb{R} \cup \{+\infty\}$, and

* This work was started during the 3rd Summerschool on Potential Theory, 2004, hosted by the College of Kecskem\'et, Faculty of Mechanical Engineering and Automation (GAMF). Both authors would like to express their gratitude for the hospitality and the support received during their stay in Kecskem\'et.

† The second named author was supported by the Hungarian Scientific Research Fund; OTKA 49448

to look at which “potential theoretic” assertions remain true in this generality (see the monograph of Landkof [12]). This approach facilitates general understanding of certain potential theoretic phenomena and allows also the exploration of fundamental principles like Frostman’s maximum principle.

Although there is a vast work done considering energy integrals and different notions of energies, the familiar notions of transfinite diameter and Chebyshev constants in this abstract setting are sporadically found, sometimes indeed inaccessible, in the literature, see Choquet [4] or Ohtsuka [17]. In [4] Choquet defines *transfinite diameter* and proves its equality with the *Wiener energy* in a rather general situation, which of course covers the classical case of the logarithmic kernel on \mathbb{C} . We give a slightly different definition for the transfinite diameter that, for infinite sets, turns out to be equivalent with the one of Choquet. The primary aim of this note is to revisit the above mentioned notions and related results and also to partly complement the theory.

We already remark here that Zaharjuta’s generalisation of transfinite diameter and Chebyshev constant to \mathbb{C}^n is completely different in nature, see [24], whereas some elementary parts of weighted potential theory (see, e.g., Mhaskar, Saff [13] and Saff, Totik [20]) could fit in this framework.

The power of the abstract potential analytic tools is well illustrated by the notion of the average distance number from metric analysis, see Gross [11], Stadje [21]. The surprising phenomenon noticed by Gross is the following: If (X, d) is a compact connected metric space, there always exists a unique number $r(X)$ (called the *average distance number* or the *rendezvous number* of X), with the property that for any finite point system $x_1, \dots, x_n \in X$ there is another point $x \in X$ with average distance

$$\frac{1}{n} \sum_{j=1}^n d(x_j, x) = r(X).$$

Stadje generalised this to arbitrary continuous, symmetric functions replacing d . Actually, it turned out, see the series of papers [6, 5, 7] and the references therein, that many of the known results concerning average distance numbers (existence, uniqueness, various generalisations, calculation techniques etc.), can be proved in a unified way using the works of Fuglede and Ohtsuka. We mention for example that Frostman’s Equilibrium Theorem is to be accounted for the existence for certain invariant measures (see Section 5 below). In these investigations the two variable versions of Chebyshev constants and energies and even their minimax duals had been needed, and were also partly available due to the works of Fuglede [10] and Ohtsuka [16, 17], see also [6].

Another occurrence of abstract Chebyshev constants is in the study of polarisation constants of normed spaces, see Anagnostopoulos, Révész [1] and Révész, Sarantopoulos [19].

Let us settle now our general framework. A *kernel* in the sense of Fuglede is a lower semicontinuous function $k : X \times X \rightarrow \mathbb{R} \cup \{+\infty\}$ [9, p. 149]. In this paper we will sometimes need that the kernel is *symmetric*, i.e., $k(x, y) = k(y, x)$. This is for example essential when defining potential and Chebyshev constant, otherwise there would be a left- and right-potential and the like.

Another assumption, however a bit of technical flavour, is the *positivity* of the kernel. This we need, because we would like to avoid technicalities when integrating not necessarily positive functions. This assumption is nevertheless not very restrictive. Since we usually consider compact sets of $X \times X$, where by lower semicontinuity k is necessarily bounded from below, we can assume that $k \geq 0$. Indeed, as we will see, energy, n^{th} diameter and n^{th} Chebyshev constant are linear in constants added to k .

Denote the set of compactly supported Radon measures on X by $\mathfrak{M}(X)$, that is

$$\begin{aligned} \mathfrak{M}(X) := \{ & \mu : \mu \text{ is a regular Borel measure on } X, \\ & \mu \text{ has compact support, } \|\mu\| < +\infty \}. \end{aligned}$$

Further, let $\mathfrak{M}_1(X)$ be the set of positive unit measures from $\mathfrak{M}(X)$,

$$\mathfrak{M}_1(X) := \{ \mu \in \mathfrak{M}(X) : \mu \geq 0, \mu(X) = 1 \}.$$

We say that $\mu \in \mathfrak{M}_1(X)$ is supported on H if $\text{supp } \mu$, which is a compact subset of X , is in H . The set of (probability) measures supported on H are denoted by $\mathfrak{M}(H)$ ($\mathfrak{M}_1(H)$).

Before recalling the relevant potential theoretic notions from [9] (see also [15]), let us spend a few words on integrals (see [2, Ch. III-IV.]). Let μ be a positive Radon measure on X . Then the integral of a compactly supported continuous function with respect to μ is the usual integral. The upper integral of a positive l.s.c. function f is defined as

$$\int_X f \, d\mu := \sup_{\substack{0 \leq h \leq f \\ h \in C_c(X)}} \int_X h \, d\mu.$$

This definition works well, because by standard arguments (see, e.g., [2, Ch. IV., Lemma 1]) one has

$$k(x, y) = \sup_{\substack{0 \leq h \leq k \\ h \in C_c(X \times X)}} h(x, y),$$

where, because of the symmetry assumption, it suffices to take only symmetric functions h in the supremum.

What should be here noted, is that this notion of integral has all useful properties that we are used to in case of Lebesgue integrals (note also the necessity of the positivity assumptions).

The usual topology on \mathfrak{M} is the so-called *vague topology* which is a locally convex topology defined by the family $\{\mu \mapsto \int_X f d\mu : f \in C_c(X)\}$ of seminorms. We will only encounter this topology in connection with families \mathcal{M} of measures supported on subsets of the same compact set $K \subset X$. In this case, the weak*-topology (determined by $C(K)$) and the vague topology coincide on \mathcal{M} , Fuglede [9].

For a potential theoretic kernel $k : X \times X \rightarrow \mathbb{R}_+ \cup \{0\}$ Fuglede [9] and Ohtsuka [15] define the *potential* and the *energy* of a measure μ

$$U^\mu(x) := \int_X k(x, y) d\mu(y), \quad W(\mu) := \iint_{X \times X} k(x, y) d\mu(y) d\mu(x).$$

The integrals exist in the above sense, although may attain $+\infty$ as well.

For a given set $H \subset X$ its *Wiener energy* is

$$w(H) := \inf_{\mu \in \mathfrak{M}_1(H)} W(\mu), \tag{1}$$

see [9, (2) on p. 153].

One also encounters the quantities (see [9, p. 153])

$$U(\mu) := \sup_{x \in X} U^\mu(x), \quad V(\mu) := \sup_{x \in \text{supp } \mu} U^\mu(x).$$

Accordingly one defines the following energy functions

$$u(H) := \inf_{\mu \in \mathfrak{M}_1(H)} U(\mu), \quad v(H) := \inf_{\mu \in \mathfrak{M}_1(H)} V(\mu).$$

In general, one has the relation

$$w \leq v \leq u \leq +\infty,$$

where in all places strict inequality may occur. Nevertheless, under our assumptions we have the equality of the energies v and w , being generally different, see [9, p. 159]. More importantly, our set of conditions suffices to have a general version of Frostman's equilibrium theorem, see Theorem 9.

In fact, at a certain point (in §4), we will also assume Frostman's maximum principle, which will trivially guarantee even $u = v$, that is, the equivalence of all three energies treated by Fuglede.

Definition. The kernel k satisfies the *maximum principle*, if for every measure $\mu \in \mathfrak{M}_1$

$$U(\mu) = V(\mu).$$

As our examples show in §5, this is essential also for the equivalence of the Chebyshev constant and the transfinite diameter. Carleson [3, Ch. III.] gives a class of examples satisfying the maximum principle: Let $\Phi(r)$, $r = |x|$, $x \in \mathbb{R}^d$ be the fundamental solution of the Laplace equation, i.e., $\Phi(|x-y|)$ the Newtonian potential on \mathbb{R}^d . For a positive, continuous, increasing, convex function H assume also that

$$\int_0^1 H(\Phi(r))r^{d-2} dr < +\infty.$$

Then $H \circ \Phi$ satisfies the maximum principle; see [3, Ch. III.] and also Fuglede [9] for further examples.

Let us now turn to the systematic treatment of the Chebyshev constant and the transfinite diameter. We call a function $g : X \rightarrow \mathbb{R}$ *log-polynomial*, if there exist $w_1, \dots, w_n \in X$ such that $g(x) = \sum_{j=1}^n k(x, w_j)$ for all $x \in X$. Accordingly, we will call the w_j s and n the zeros and the degree of $g(x)$, respectively. Obviously the sum of two log-polynomials is a log-polynomial again. The terminology here is motivated by the case of the logarithmic kernel

$$k(x, y) = -\log|x - y|,$$

where the log-polynomials correspond to negative logarithms of algebraic polynomials.

Log-polynomials give access to the definition of *transfinite diameter* and the *Chebyshev constant*, see Carleson [3], Choquet [4], Fekete [8], Ohtsuka [17] and Pólya, Szegő [18]. First we start with the “degree n ” versions, whose convergence will be proved later.

Definition. Let $H \subset X$ be fixed. We define the n^{th} diameter of H as

$$D_n(H) := \inf_{w_1, \dots, w_n \in H} \frac{1}{(n-1)n} \left(\sum_{1 \leq j \neq l \leq n} k(w_j, w_l) \right); \quad (2)$$

or, if the kernel is symmetric

$$D_n(H) = \inf_{w_1, \dots, w_n \in H} \frac{2}{(n-1)n} \left(\sum_{1 \leq i < j \leq n} k(w_i, w_j) \right).$$

If H is compact, then due to the fact that k is l.s.c., $D_n(H)$ is attained for some points $w_1, \dots, w_n \in H$, which are then called *n-Fekete points*. We will also use the term *approximate n-Fekete points* with the obvious meaning. Note also that for a finite set H , $\#H = m$ and $n > m$, there is always a point from the diagonal $\Delta = \{(x, x) : x \in H\}$ in the definition of $D_n(H)$. This possibility is completely excluded by Choquet in [4], thus allowing only infinite sets.

Definition. For an arbitrary $H \subset X$ the n^{th} Chebyshev constant of H is defined as

$$M_n(H) := \sup_{w_1, \dots, w_n \in H} \inf_{x \in H} \frac{1}{n} \left(\sum_{k=1}^n k(x, w_k) \right)$$

We are going to show that both n^{th} diameters and n^{th} Chebyshev constants converge from below to some number (or $+\infty$), which are respectively called the *transfinite diameter* $D(H)$ and the *Chebyshev constant* $M(H)$. The aim of this paper is to relate these quantities as well as the Wiener energy of a set.

2. Chebyshev constant and transfinite diameter

We define the Chebyshev constant and the transfinite diameter of a set $H \subset X$ and proceed analogously to the classical case. It turns out, though not very surprisingly, that in general the equality of these two quantities does not hold.

First, we prove the convergence of n^{th} diameters and n^{th} Chebyshev constants. This is for both cases classical, we give the proof only for the sake of completeness, see, e.g., Carleson [3], Choquet [4], Fekete [8], Ohtsuka [17] and Pólya, Szegő [18].

PROPOSITION 1. *The sequence of n^{th} diameters is monotonically increasing.*

Proof. Choose $x_1, \dots, x_n \in H$ arbitrarily. If we leave out any index $i = 1, 2, \dots, n$, then for the remaining $n - 1$ points we obtain by the definition of $D_{n-1}(H)$ that

$$\frac{1}{(n-1)(n-2)} \sum_{\substack{1 \leq j \neq l \leq n \\ j \neq i, l \neq i}} k(x_j, x_l) \geq D_{n-1}(H).$$

After summing up for $i = 1, 2, \dots, n$ this yields

$$\frac{1}{n-1} \sum_{1 \leq j \neq l \leq n} k(x_j, x_l) \geq n \cdot D_{n-1}(H),$$

for each term $k(x_j, x_l)$ occurs exactly $n - 2$ times. Now taking the infimum for all possible $x_1, \dots, x_n \in H$, we obtain $n \cdot D_n(H) \geq n \cdot D_{n-1}(H)$, hence the assertion. \blacksquare

The limit $D(H) := \lim_{n \rightarrow \infty} D_n(H)$ is the *transfinite diameter* of H .

Similarly, the n^{th} Chebyshev constants converge, too.

PROPOSITION 2. *For any $H \subset X$, the Chebyshev constants $M_n(H)$ converge in the extended sense.*

Proof. The sum of two log-polynomials, $p(z) = \sum_{i=1}^n k(z, x_i)$ with degree n and $q(z) = \sum_{j=1}^m k(z, y_j)$ with degree m , is also a log-polynomial with degree $n + m$. Therefore

$$(n+m)M_{n+m} \geq nM_n + mM_m \tag{3}$$

for all n, m follows at once. Should $M_n(H)$ be infinity for some n , then all succeeding terms $M_{n'}(H)$, $n' \geq n$ are infinity as well, hence the convergence is obvious. We assume now that $M_n(H)$ is a finite sequence. At this point, for the sake of completeness, we can repeat the classical argument of Fekete [8].

Namely, let m, n be fixed integers. Then there exist $l = l(n, m)$ and $r = r(n, m)$, $0 \leq r < m$ nonnegative integers such that $n = l \cdot m + r$. Iterating the previous inequality (3) we get

$$n \cdot M_n \geq l(mM_m) + rM_r = nM_m + r(M_r - M_m).$$

Fixing now the value of m , the possible values of r remain bounded by m , and the finitely many values of $M_r - M_m$'s are finite, too. Hence dividing both sides by n , and taking $\liminf_{n \rightarrow \infty}$, we are led to

$$\liminf_{n \rightarrow \infty} M_n \geq \liminf_{n \rightarrow \infty} \left(M_m + \frac{r}{n} (M_r - M_m) \right) = M_m .$$

This holds for any fixed $m \in \mathbb{N}$, so taking $\limsup_{m \rightarrow \infty}$ on the right hand side we obtain

$$\liminf_{n \rightarrow \infty} M_n \geq \limsup_{m \rightarrow \infty} M_m,$$

that is, the limit exists. \blacksquare

$M(H) := \lim_{n \rightarrow \infty} M_n(H)$ is called the *Chebyshev constant* of H .

In the following, we investigate the connection between the Chebyshev constant $M(H)$ and the transfinite diameter $D(H)$.

THEOREM 3. *Let k be a positive, symmetric kernel. For any $n \in \mathbb{N}$ and $H \subset X$ we have $D_n(H) \leq M_n(H)$, thus also $D(H) \leq M(H)$.*

Proof. If $M_n(H) = +\infty$, then the assertion is trivial. So assume $M_n(H) < +\infty$. By the quasi-monotonicity (see (3)) we have that for all $m \leq n$ also $M_m(H)$ is finite. We use this fact to recursively find $w_1, \dots, w_n \in H$ such that $k(w_i, w_j) < +\infty$ for all $i < j \leq n$. At the end we arrive at $\sum_{1 \leq i < j \leq n} k(w_i, w_j) < +\infty$, hence $D_n(H) < +\infty$. This was our first aim to show, in the following this choice of the points w_1, \dots, w_n will not play any role. Instead, for an arbitrarily fixed $\varepsilon > 0$, we take, as we may, an “approximate n -Fekete point system” w_1, \dots, w_n with

$$\frac{1}{(n-1)n} \sum_{1 \leq i \neq j \leq n} k(w_i, w_j) < D_n + \varepsilon. \quad (4)$$

For any $x \in H$ the points x, w_1, \dots, w_n form a point system of $n+1$ points, so by the definition of D_{n+1} we have

$$2 \sum_{i=1}^n k(x, w_i) + \sum_{1 \leq i \neq j \leq n} k(w_i, w_j) \geq n(n+1)D_{n+1} \geq n(n+1)D_n,$$

using also the monotonicity of the sequence D_n . This together with (4) lead to

$$p_n(x) := \sum_{i=1}^n k(x, w_i) \geq \frac{n(n+1)}{2} D_n - \frac{n(n-1)}{2} (D_n + \varepsilon).$$

Taking infimum of the left hand side for $x \in H$ we obtain

$$\inf_{x \in H} p_n(x) \geq nD_n - \frac{n(n-1)\varepsilon}{2}.$$

By the very definition of the n^{th} Chebyshev constant, $n \cdot M_n \geq \inf_{x \in H} p_n(x)$ holds, hence $M_n \geq D_n - (n-1)\varepsilon/2$ follows. As this holds for all $\varepsilon > 0$, we conclude $M_n \geq D_n$. ■

Later we will show that, unlike the classical case of \mathbb{C} , the strict inequality $D < M$ is well possible.

3. Transfinite diameter and energy

We study the connection between the energy w and the transfinite diameter D . Without assuming the maximum principle we can prove the equivalence of these two quantities for compact sets. This result can actually be found in a note of Choquet [4]. There is however a

slight difference to the definitions of Choquet in [4]. There the diagonal was completely excluded from the definition of D , that is the infimum in (2) is taken over $w_i \neq w_j$, $i \neq j$ and not for systems of arbitrary w_j 's. This means, among others, that in [4] the transfinite diameter is only defined for infinite sets. The other assumption of Choquet is that the kernel is infinite on the diagonal. This is completely the contrary to what we assume in Theorem 8. Indeed, with our definitions of the transfinite diameter one can even prove equality for arbitrary sets if the kernel is finite-valued.

THEOREM 4. *Let k be an arbitrary kernel and $H \subset X$ be any set. Then $D(H) \leq w(H)$.*

Proof. Let $\mu \in \mathfrak{M}_1(H)$ be arbitrary, and define $\nu := \bigotimes_{j=1}^n \mu$ the product measure on the product space X^n . We can assume that the kernel is positive because $\text{supp } \mu$, and hence $\text{supp } \nu$, is compact so we can add a constant to k such that it will be positive on these supports. Consider the following lower semicontinuous functions g and h on X^n

$$\begin{aligned} g : (x_1, \dots, x_n) &\mapsto D_n(H) \left(:= \inf_{(w_1, \dots, w_n) \in X^n} \frac{1}{n(n-1)} \sum_{1 \leq i \neq j \leq n} k(w_i, w_j) \right) \\ h : (x_1, \dots, x_n) &\mapsto \frac{1}{n(n-1)} \sum_{1 \leq i \neq j \leq n} k(x_i, x_j). \end{aligned}$$

Since $0 \leq g \leq h$, by the definition of the upper integral the following holds true

$$\begin{aligned} D_n(H) &\leq \int_{X^n} \frac{1}{n(n-1)} \sum_{1 \leq i \neq j \leq n} k(x_i, x_j) d\nu(x_1, \dots, x_n) \\ &= \frac{1}{n(n-1)} \sum_{1 \leq i \neq j \leq n} \int_{H^2} k(x_i, x_j) d\mu(x_i) d\mu(x_j) = W(\mu). \end{aligned}$$

Taking infimum in μ yields $D_n(H) \leq w(H)$, hence also $D(H) \leq w(H)$. ■

To establish the converse inequality we need a compactness assumption. With the slightly different terminology, Choquet proves the following for kernels being $+\infty$ on the diagonal Δ . The arguments there are very similar, except that the diagonal doesn't have to be taken care of in [4]. We give a detailed proof.

PROPOSITION 5 (Choquet [4]). *For an arbitrary kernel function k the inequality $D(K) \geq w(K)$ holds for all $K \subseteq X$ compact sets.*

Proof. First of all the l.s.c. function k attains its infimum on the compact set $K \times K$. So by shifting k up we can assume that it is

positive, and the validity of the desired inequality is not influenced by this.

If $D(K) = +\infty$, then by Theorem 4 we have $w(K) = +\infty$, thus the assertion follows. Assume therefore $D(K) < +\infty$, and let $n \in \mathbb{N}$, $\varepsilon > 0$ be fixed. Let us choose a Fekete point system w_1, \dots, w_n from K . Put $\mu := \mu_n := 1/n \sum_{i=1}^n \delta_{w_i}$ where δ_{w_i} are the Dirac measures at the points w_i , $i = 1, \dots, n$. For a continuous function $0 \leq h \leq k$ with compact support, we have

$$\begin{aligned} \iint_{K \times K} h \, d\mu \, d\mu &= \frac{1}{n^2} \sum_{i,j=1}^n h(w_i, w_j) \\ &= \frac{1}{n^2} \sum_{i=1}^n h(w_i, w_i) + \frac{1}{n^2} \sum_{\substack{i,j=1 \\ i \neq j}}^n h(w_i, w_j) \\ &\leq \frac{1}{n^2} \sum_{i=1}^n h(w_i, w_i) + \frac{1}{n^2} \sum_{\substack{i,j=1 \\ i \neq j}}^n k(w_i, w_j) \\ &\leq \frac{\|h\|}{n} + \frac{1}{n^2} \sum_{\substack{i,j=1 \\ i \neq j}}^n k(w_i, w_j) \\ &\leq \frac{\|h\|}{n} + \frac{n-1}{n} D_n(K) \leq \frac{\|h\|}{n} + D(K) \end{aligned}$$

using, in the last step, also the monotonicity of the sequence D_n (Proposition 1). In fact, we obtain for $n \geq N = N(\|h\|, \varepsilon)$ the inequality

$$\iint_{K \times K} h \, d\mu \, d\mu \leq D + \varepsilon. \quad (5)$$

It is known, essentially by the Banach-Alaoglu Theorem, that for a compact set K the measures of $\mathfrak{M}_1(K)$ form a weak*-compact subset of \mathfrak{M} , hence there is a cluster point $\nu \in \mathfrak{M}_1(K)$ of the set $\mathcal{M}_N := \{\mu_n : n \geq N\} \subset \mathfrak{M}_1(K)$. Let $\{\nu_\alpha\}_{\alpha \in I} \subseteq \mathcal{M}_N$ be a net converging to ν . Recall that $\nu_\alpha \otimes \nu_\alpha$ weak*-converges to $\nu \otimes \nu$. We give the proof. For a function $g \in C(K \times K)$, $g(x, y) = g_1(x) \cdot g_2(y)$ it is obvious that

$$\iint_{K \times K} g \, d\nu_\alpha \, d\nu_\alpha \rightarrow \iint_{K \times K} g \, d\nu \, d\nu. \quad (6)$$

The set \mathcal{A} of such product-decomposable functions $g(x, y) = g_1(x)g_2(y)$ is a subalgebra of $C(K \times K)$, which also separates $X \times X$, since it is already coordinatewise separating. By the Stone–Weierstraß theorem \mathcal{A} is dense in $C(K \times K)$. From this, using also that the family \mathcal{M}_N of

measures is norm-bounded, we immediately get the weak*-convergence (6). All these imply

$$\iint_{K \times K} h \, d\nu \, d\nu \leq D(K) + \varepsilon,$$

thus

$$w(K) \leq W(\nu) := \iint_{K \times K} k \, d\nu \, d\nu = \sup_{\substack{0 \leq h \leq k \\ h \in C_c(X \times X)}} \iint_{K \times K} h \, d\nu \, d\nu \leq D(K) + \varepsilon,$$

for all $\varepsilon > 0$. This shows $w(K) \leq D(K)$. \blacksquare

COROLLARY 6 (Choquet [4]). *For arbitrary kernel k and compact set $K \subset X$, the equality $D(K) = w(K)$ holds.*

Proof. By compactness we can shift k up and therefore assume it is positive. Then we apply Theorem 4 and Proposition 5. \blacksquare

The assumptions of Choquet [4] are the compactness of the set plus the property that the kernel is $+\infty$ on the diagonal (besides it is continuous in the extended sense). This ensures, loosely speaking, that for a set K of finite energy an energy minimising measure μ (i.e., for which $W(\mu) = w(K)$) is necessarily non-atomic, moreover $\mu \otimes \mu$ is not concentrated on the diagonal. Therefore to show equality of w with D , one has to exclude the diagonal completely from the definition of the transfinite diameter.

We however allow a larger set of choices for the point system in the definition of D . Indeed, we allow Fekete points to coincide, and this also makes it possible to define the transfinite diameter of finite sets. With this setup the inequality $D \leq w$ is only simpler than in the case handled by Choquet. Whereas, however surprisingly, the equality $D(K) = w(K)$ is still true for compact sets K but without the assumption on the diagonal values of the kernel.

We will see in §5 Example 13 that even assuming the maximum principle but lacking the compactness allows the strict inequality $D < w$. This phenomena however may exist only in case of unbounded kernels, as we will see below. In fact, we show that if the kernel is finite on the diagonal, then $D = w$ holds for arbitrary sets. For this purpose, we need the following technical lemma, which shows certain inner regularity properties of D and is also interesting in itself.

LEMMA 7. *Assume that the kernel k is positive and finite on the diagonal, i.e., $k(x, x) < +\infty$ for all $x \in X$. Then for an arbitrary*

$H \subset X$ we have

$$D(H) = \inf_{\substack{K \subset H \\ K \text{ compact}}} D(K) = \inf_{\substack{W \subset H \\ \#W < \infty}} D(W). \quad (7)$$

Proof. The inequality $\inf D(K) \leq \inf D(W)$ is clear. For $H \supseteq K$ the inequality $D(H) \leq D(K)$ is obvious, so we can assume $D(H) < +\infty$. For $\varepsilon > 0$ let $W = \{w_1, \dots, w_n\}$ be an approximate n -Fekete point set of H satisfying (4). Then

$$D(W) = \lim_{m \rightarrow \infty} D_{mn}(W) \leq \lim_{m \rightarrow \infty} \frac{1}{mn(mn-1)} \sum_{1 \leq i' \neq j' \leq mn} k(w_{i'}, w_{j'}),$$

where

$$w_{i'} := \begin{cases} \dots & i' = i + rn, \quad r = 0, \dots, m-1 \\ w_i \\ \dots \end{cases}$$

Set $C := \max\{k(x, x) : x \in W\}$. So we find

$$\begin{aligned} D(W) &\leq \lim_{m \rightarrow \infty} \left\{ \frac{m^2}{mn(mn-1)} \sum_{1 \leq i \neq j \leq n} k(w_i, w_j) + \frac{m-1}{mn(mn-1)} \sum_{1 \leq i \leq n} k(w_i, w_i) \right\} \\ &\leq \sum_{1 \leq i \neq j \leq n} k(w_i, w_j) \lim_{m \rightarrow \infty} \frac{m^2}{mn(mn-1)} + Cn \lim_{m \rightarrow \infty} \frac{m-1}{mn(mn-1)} \\ &= \frac{1}{n^2} \sum_{1 \leq i \neq j \leq n} k(w_i, w_j) \leq \frac{n-1}{n} (D_n(H) + \varepsilon) \leq D(H) + \varepsilon. \end{aligned}$$

This being true for all $\varepsilon > 0$, taking infimum we finally obtain

$$\inf_{\substack{W \subset H \\ \#W < \infty}} D(W) \leq D(H).$$

■

Clearly, if $k(x, x) = +\infty$ for all $x \in W$ with a finite set $\#W = n$, then for all $m > n$ we have $D_m(W) = +\infty$. Thus in particular for kernels with $k : \Delta \rightarrow \{+\infty\}$, the above can not hold in general, at least as regards the last part with finite subsets.

Now, completely contrary to Choquet [4] we assume that the kernel is finite on the diagonal and prove $D = w$ for any set. Hence an example of $D < w$ (see §5 Example 13) must assume $k(x, x) = +\infty$ at least for some point x .

THEOREM 8. *Assume that the kernel k is positive and is finite on the diagonal, that is $k(x, x) < +\infty$ for all $x \in X$. Then for arbitrary sets $H \subset X$, the equality $D(H) = w(H)$ holds.*

Proof. By Theorem 4 we have $D(H) \leq w(H)$. Hence there is nothing to prove, if $D(H) = +\infty$. Assume $D(H) < +\infty$, and let $\varepsilon > 0$ be arbitrary. By Lemma 7 we have for some $n \in \mathbb{N}$ a finite set $W = \{w_1, w_2, \dots, w_n\}$ with $D(H) + \varepsilon \geq D(W)$. In view of Proposition 5 we have $D(W) \geq w(W)$, and by monotonicity also $w(W) \geq w(H)$. It follows that $D(H) + \varepsilon \geq w(H)$ for all $\varepsilon > 0$, hence also the “ \geq ” part of the assertion follows. ■

4. Energy and Chebyshev constant

To investigate the relationship between the energy and the Chebyshev constant the following general version of Frostman’s Equilibrium Theorem [9, Theorem 2.4] is fundamental for us.

THEOREM 9 (Fuglede). *Let k be a positive, symmetric kernel and $K \subset X$ be a compact set such that $w(K) < +\infty$. Every μ which has minimal energy ($\mu \in \mathfrak{M}_1(K)$, $W(\mu) = w(K)$) satisfy the following properties*

$$\begin{aligned} U^\mu(x) &\geq w(K) \quad \text{for nearly every}^1 x \in K, \\ U^\mu(x) &\leq w(K) \quad \text{for every } x \in \text{supp } \mu, \\ U^\mu(x) &= w(K) \quad \text{for } \mu\text{-almost every } x \in X. \end{aligned}$$

Moreover, if the kernel is continuous, then

$$U^\mu(x) \geq w(K) \quad \text{for every } x \in K.$$

THEOREM 10. *Let $H \subset X$ be arbitrary. Assume that the kernel k is positive, symmetric and satisfies the maximum principle. Then we have $M_n(H) \leq w(H)$ for all $n \in \mathbb{N}$, whence also $M(H) \leq w(H)$ holds true.*

Proof. Let $n \in \mathbb{N}$ be arbitrary. First let K be any compact set. We can assume $w(K) < +\infty$, since otherwise the inequality holds irrespective of the value of $M_n(K)$. Consider now an energy-minimising measure ν_K of K , whose existence is assured by the lower semicontinuity of $\mu \mapsto \iint k \, d\mu \, d\mu$ and the compactness of $\mathfrak{M}_1(K)$, see [9, Theorem 2.3].

By the Frostman-Fuglede theorem (Theorem 9) we have $U^{\nu_K}(x) \leq w(K)$ for all $x \in \text{supp } \nu_K$, so $V(\nu_K) \leq w(K)$, and by the maximum principle even

$$U^{\nu_K}(x) \leq w(K) \quad \text{for all } x \in X.$$

¹ The set A of exceptional points is small in the sense $w(A) = +\infty$.

Then for all $w_1, \dots, w_n \in K$

$$\inf_{x \in K} \frac{1}{n} \sum_{j=1}^n k(x, w_j) \leq \int_X \frac{1}{n} \sum_{j=1}^n k(x, w_j) d\nu_K(x) \leq w(K).$$

Taking supremum for $w_1, \dots, w_n \in K$, we obtain

$$\sup_{w_1, \dots, w_n \in K} \inf_{x \in K} \frac{1}{n} \sum_{j=1}^n k(x, w_j) \leq w(K).$$

So $M_n(K) \leq w(K)$ for all $n \in \mathbb{N}$.

Next let $H \subset X$ be arbitrary. In view of the last form of (1), for all $\varepsilon > 0$ there exists a measure $\mu \in \mathfrak{M}_1(H)$, compactly supported in H , with $w(\mu) \leq w(H) + \varepsilon$. Let $W = \{w_1, \dots, w_n\} \subset H$ be arbitrary and define $p_W(x) := \frac{1}{n} \sum_i k(x, w_i)$.

Consider the compact set $K := W \cup \text{supp } \mu \subset H$. By definition of the energy, $\text{supp } \mu \subset K$ implies $w(K) \leq w(\mu)$, hence $w(K) \leq w(H) + \varepsilon$. Combining this with the above, we come to $M_n(K) \leq w(H) + \varepsilon$. Since $W \subset K$, by definition of $M_n(K)$ we also have

$$\inf_{x \in K} p_W(x) \leq M_n(K). \quad (8)$$

The left hand side does not increase, if we extend the inf over the whole of H , and the right hand side is already estimated from above by $w(H) + \varepsilon$. Thus (8) leads to

$$\inf_{x \in H} p_W(x) \leq w(H) + \varepsilon.$$

This holds for all possible choices of $W = \{w_1, \dots, w_n\} \subset H$, hence is true also for the sup of the left hand side. By definition of $M_n(H)$ this gives exactly $M_n(H) \leq w(H) + \varepsilon$, which shows even $M_n(H) \leq w(H)$. ■

Remark. In [6] it is proved that $M(H) = q(H)$, where

$$q(H) = \inf_{\mu \in \mathfrak{M}_1(H)} \sup_{x \in H} U^\mu(x).$$

The idea behind is a minimax theorem, see also [16, 17]. Trivially $w(H) \leq q(H) \leq u(H)$. So the maximum principle implies $M(H) = w(H) = q(H) = u(H)$.

5. Summary of the Results. Examples

In this section, we put together the previous results, thus proving the equality of the three quantities being studied, under the assumption of the maximum principle for the kernel. Further, via several instructive examples we investigate the necessity of our assumptions and the sharpness of the results.

THEOREM 11. *Assume that the kernel k is positive, symmetric and satisfies the maximum principle. Let $K \subset X$ be any compact set. Then the transfinite diameter, the Chebyshev constant and the energy of K coincide:*

$$D(K) = M(K) = w(K).$$

Proof. We presented a cyclic proof above, consisting of $M \geq D$ (Theorem 3), $D \geq w$ (Proposition 5) and finally $w \geq M$ (Theorem 10). \blacksquare

THEOREM 12. *Assume that the kernel k is positive, finite and satisfies the maximum principle. For an arbitrary subset $H \subset X$ the transfinite diameter, the Chebyshev constant and the energy of H coincide:*

$$D(H) = M(H) = w(H).$$

Proof. By finiteness $D = w$, due to Theorem 8. This with $D \leq M$ and $M \leq w$ (Theorems 3 and 10) proves the assertion. \blacksquare

Remark. In the above theorem, logically it would suffice to assume that the kernel be finite only on the diagonal. But if this was the case, the maximum principle would then immediately imply the finiteness of the kernel *everywhere*.

Let us now discuss how sharp the results of the preceding sections are. In the first example we show that, if we drop the assumption of compactness the assertions of Theorem 3, Theorem 4 and Theorem 10 are in general the strongest possible.

Example 13. Let $X = \mathbb{N} \cup \{0\}$ endowed with discrete topology and the kernel

$$k(n, m) := \begin{cases} +\infty & \text{if } n = m, \\ 0 & \text{if } 0 \neq n \neq m \neq 0, \\ 1 & \text{otherwise.} \end{cases}$$

The kernel is symmetric, l.s.c. and has the maximum principle. This latter can be seen by noticing that for a probability measure $\mu \in \mathfrak{M}_1(X)$

the potential is $+\infty$ on the support of μ . Indeed, since X is countable, all measures $\mu \in \mathfrak{M}_1(X)$ are necessarily atomic, and if for some point $\ell \in X$ we have $\mu(\{\ell\}) > 0$, then by definition $\int_X k(x, y) d\mu(y) = +\infty$.

We calculate the studied quantities of the set $H = X$ (also as in all the examples below). Since the kernel is positive, $D_n \geq 0$. On the other hand, choosing $w_1 := 1, \dots, w_n := n$, all the values $k(w_i, w_j)$ will be exactly 0, so it follows that $D_n = 0$, $n = 1, 2, \dots$, and hence $D = 0$.

The Chebyshev constant can be estimated from below, if we compute the infimum of a suitably chosen log-polynomial. Consider the log-polynomial $p(x)$ with all zeros placed at 0, that is with $w_1 = \dots = w_n = 0$. Then the log-polynomial $p(x)$ is $\sum_j k(x, w_j) = n \cdot k(x, 0)$. If $x \neq 0$, we have $p(x) = n$, which gives $M \geq 1$. The upper estimate of M is also easy: suppose that in the system w_1, \dots, w_n there are exactly m points being equal to 0 (say the first m). Then

$$p(x) = \begin{cases} +\infty & x = w_1, \dots, w_n, \\ n & x = 0, x \neq w_1, \dots, w_n \\ m & x \neq 0, x \neq w_1, \dots, w_n \end{cases} \quad (\text{if } m = 0)$$

This shows for the corresponding log-polynomial $\inf p(x) = m$, so $M_n \leq 1$, whence $M = 1$.

The energy is computed easily. Using the above reasoning on the maximum principle, we see $W(\mu) = +\infty$ for any $\mu \in \mathfrak{M}_1(X)$, hence $w(X) = +\infty$.

Thus we have an example of

$$+\infty = w > M > D = 0.$$

The above example completes the case of the kernel with maximum principle. Let us now drop this assumption and look at what can happen.

Example 14. Let $X := \{-1, 0, 1\}$ be endowed with the discrete topology. We define the kernel by

$$k(x, y) := \begin{cases} 2 & \text{if } 0 \leq |x - y| < 2, \\ 0 & \text{if } 2 = |x - y|. \end{cases}$$

Then k is continuous and bounded on $X \times X$. This, in any case, implies $D = w$ by Theorem 8. Note that k does not satisfy the maximum principle. To see this, consider, e.g., the measure $\mu = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$. Then for the potential U^μ one has $U^\mu(1) = U^\mu(-1) = 1$ and $U^\mu(0) = 2$, which shows the failure of the maximum principle.

To estimate the n^{th} diameter from above, let us consider the point system $\{w_i\}$ of $n = 2m$ points with m points falling at -1 and m points

falling at 1, while no points being placed at 0. Then by definition of $D_n := D_n(X)$ one can write

$$\frac{n(n-1)}{2}D_n \leq 2 \binom{m}{2} \cdot 2 + m^2 \cdot 0 = \frac{n^2}{2} - n.$$

Applying this estimate for all even $n = 2m$ as $n \rightarrow \infty$, it follows that

$$D = \lim_{n \rightarrow \infty} D_n \leq 1. \quad (9)$$

Next we estimate the Chebyshev constants from below by computing the infimum of some special log-polynomials. For $p_n(x) = k(x, 0)$ one has $p_n(x) \equiv 2 = \inf p_n$. We thus find $M_n \geq 2$ and $M \geq 2$, showing $M > D$, as desired.

Example 15. Let $X := \mathbb{N}$ with the discrete topology. Then X is a locally compact Hausdorff space, and all functions are continuous, hence l.s.c. on X . Let $k : X \times X \rightarrow [0, +\infty]$ be defined as

$$k(n, m) := \begin{cases} +\infty & \text{if } n = m, \\ 2^{-n-m} & \text{if } n \neq m. \end{cases}$$

Clearly k is an admissible kernel function. For the energy we have again $w(X) = +\infty$, see Example 13.

On the other hand let $n \in \mathbb{N}$ be any fixed number, and compute the n^{th} diameter $D_n(X)$. Clearly if we choose $w_j := m + j$, with m a given (large) number to be chosen, then we get

$$D_n(H) \leq \frac{1}{(n-1)n} \sum_{1 \leq i \neq j \leq n} 2^{-i-j-2m} \leq \frac{2^{-2m}}{(n-1)n} \left(\sum_{i=1}^{\infty} 2^{-i} \right)^2 \leq 2^{-2m},$$

hence we find that the n^{th} diameter is $D_n(X) = 0$, so $D(X) = 0$, too. For any log-polynomial $p(x)$ we have $\inf p(x) = \lim_{x \rightarrow \infty} p(x) = 0$, hence $M(X) = 0$. That is we have $D(X) = M(X) = 0 < w(X) = +\infty$.

The example shows how important the diagonal, excluded in the definition of D but taken into account in w , may become for particular cases. We can even modify the above example to get finite energy.

Example 16. Let $X := (0, 1]$, equipped with the usual topology, and let $x_n = 1/n$. We take now

$$k(x, y) := \begin{cases} +\infty & \text{if } x = y, \\ 2^{-n-m} & \text{if } x = x_n \text{ and } y = x_m \ (x_n \neq x_m), \\ -\log|x - y| & \text{otherwise} \end{cases}$$

Compared to the l.s.c. logarithmic kernel, this k assumes different, smaller values at the relatively closed set of points $\{(x_n, x_m) : n \neq m\} \subset X \times X$ only, hence it is also l.s.c. and thus admissible as kernel.

If a measure $\mu \in \mathfrak{M}_1(X)$ has any atom, say if for some point $z \in X$ we have $\mu(\{z\}) > 0$, then by definition $\int_X k(x, y) d\mu(y) = +\infty$, hence also $w(\mu) = +\infty$. Since for all $\mu \in \mathfrak{M}_1(X)$ with any atomic component $w(\mu) = +\infty$, we find that for the set $H := X$ we have

$$w(H) := \inf_{\mu \in \mathfrak{M}_1(H)} w(\mu) = \inf_{\substack{\mu \in \mathfrak{M}_1(H) \\ \mu \text{ not atomic}}} w(\mu).$$

But for measures without atoms, the countable set of the points x_n are just of measure zero, hence the energy equals to the energy with respect to the logarithmic kernel. Thus we conclude $w(H) = e^{-\text{cap}(H)} = e^{-1/4}$, as $\text{cap}((0, 1]) = 1/4$ is well-known.

On the other hand if $n \in \mathbb{N}$ is any fixed number, we can compute the n^{th} diameter $D_n(H)$ exactly as above in Example 15. Hence it is easy to see that $D_n(H) = 0$, whence also $D(H) = 0$. Similarly, we find $M(H) = 0$, too.

This example shows that even in case $w(H) < +\infty$ we can have $w(H) > D(H) = M(H)$.

6. Average distance number and the maximum principle

In the previous section, we showed the equality of the Chebyshev constant M and the transfinite diameter D , using essentially elementary inequalities and the only theoretically deeper ingredient, the assumption of the maximum principle. We have also seen examples showing that the lack of the maximum principle for the kernel allows strict inequality between M and D . These observations certify to the relevance of this principle in our investigations. Indeed, in this section we show the necessity of the maximum principle in case of continuous kernels for having $M(K) = D(K)$ for all compact sets K . We need some preparation first.

Recall from the introduction the notion of the average distance (or rendezvous) number. Actulyally, a more general assertion than there can be stated, see Stadje [21] or [6]. For a compact connected set K and a continuous, symmetric kernel k , the average distance number $r(K)$ is the uniquely existing number with the property that for all probability measures supported in K there is a point $x \in K$ with

$$U^\mu(x) = \int_K k(x, y) d\mu(y) = r(K).$$

This can be even further generalised by dropping the connectedness, see Thomassen [22] and [6]. Even for not necessarily connected but compact spaces K with symmetric, continuous kernel k there is a unique number $r(K)$ with the property that whenever a probability measure on K and a positive ε are given, there are points $x_1, x_2 \in K$ such that

$$U^\mu(x_1) - \varepsilon \leq r(K) \leq U^\mu(x_2) + \varepsilon.$$

This number is called the (*weak*) *average distance number*, and is particularly easy to calculate, when a probability measure with constant potential is available. Such a measure μ is called then an *invariant measure*. In this case the average distance number $r(K)$ is trivially just the constant value of the potential U^μ , see Morris, Nicholas [14] or [7].

It was proved in [7] that one always has $M(K) = r(K)$, so once we have an invariant measure, then the Chebyshev constant is again easy to determine.

Also the Wiener energy $w(K)$ has connection to invariant measures, as shown by the following result, which is a simplified version of a more general statement from [7], see also Wolf [23].

THEOREM 17. *Let $\emptyset \neq K \subset X$ be a compact set and k be a continuous, symmetric kernel. Then we have*

$$r(K) \geq w(K).$$

Furthermore, if $r(K) = w(K)$, then there exists an invariant measure in $\mathfrak{M}_1(K)$.

As mentioned above, we have $r(K) = M(K)$, so the inequality $r(K) \geq w(K)$ in the first assertion of the above theorem is also the consequence of Theorems 3 and 8. For the proof of the second assertion one can use the Frostman-Fuglede Equilibrium Theorem 9 with the obvious observation that “nearly every” in this context means indeed “every”. Actually any probability measure $\mu \in \mathfrak{M}_1(K)$ which minimises $\nu \mapsto \sup_K U^\nu$ is an invariant measure and its potential is constant $M(K)$, see [7, Thm. 5.2] (such measures undoubtedly exist because of compactness of $\mathfrak{M}_1(K)$). Henceforth we will indifferently use the terms energy minimising or invariant for expressing this property of measures.

THEOREM 18. *Suppose that the kernel k is symmetric and continuous. If $M(K) = D(K)$ for all compact sets $K \subseteq X$, then the kernel has the maximum principle.*

Proof. Recall from Corollary 6 that $D(K) = w(K)$ for all $K \subseteq X$ compact. So we can use Theorem 17 all over in the following arguments. We first prove the assertion in the case when X is a finite set. The

proof is by induction on $n = \#X$. For $n = 1$ the assertion is trivial. Let now $\#X = 2$, $X = \{a, b\}$. Assume without loss of generality that $k(a, a) \leq k(b, b)$. Then we only have to prove that for $\mu = \delta_a$ the maximum principle, i.e., the inequality $k(a, b) \leq k(a, a)$ holds. To see this we calculate $M(X)$ and $D(X)$. We certainly have $D(X) \leq k(a, a)$. On the other hand for an energy minimising probability measure $\nu_p := p\delta_a + (1-p)\delta_b$ on X we know that its potential is constant over X , hence

$$\begin{aligned} pk(a, a) + (1-p)k(b, a) &= pk(a, b) + (1-p)k(b, b) \\ &= M(X) = D(X) \leq k(a, a). \end{aligned}$$

Here if $p = 1$, then $k(a, a) = k(a, b)$. If $p < 1$, then we can write

$$(1-p)k(b, a) \leq (1-p)k(a, a), \quad \text{hence } k(b, a) \leq k(a, a),$$

so the maximum principle holds.

Assume now that the assertion is true for all sets with at most n elements and for all kernels, and let $\#X = n + 1$. For a probability measure μ on X we have to prove $\sup_{x \in X} U^\mu(x) = \sup_{x \in \text{supp } \mu} U^\mu(x)$. If $\text{supp } \mu = X$, then there is nothing to prove. Similarly, if there are two distinct points $x_1 \neq x_2$, $x_1, x_2 \in X \setminus \text{supp } \mu$, then by the induction hypothesis we have

$$\sup_{x \in X \setminus \{x_1\}} U^\mu(x) = \sup_{x \in \text{supp } \mu} U^\mu(x) = \sup_{x \in X \setminus \{x_2\}} U^\mu(x).$$

So for a probability measure μ defying the maximum principle we must have $\#\text{supp } \mu = n$, say $\text{supp } \mu = X \setminus \{x_{n+1}\}$; let μ be such a measure. Set $K = \text{supp } \mu$ and let μ' be an invariant measure on K . We claim that all such measures μ' are also violating the maximum principle. If $\mu = \mu'$, we are done. Assume $\mu \neq \mu'$ and consider the linear combinations $\mu_t := t\mu + (1-t)\mu'$. There is a $\tau > 1$, for which μ_τ is still a probability measure and $\text{supp } \mu_\tau \subsetneq \text{supp } \mu$. By the inductive hypothesis (as $\#\text{supp } \mu_\tau < n$) we have $U^{\mu_\tau}(x_{n+1}) \leq U^{\mu_\tau}(a)$ for some $a \in \text{supp } \mu_\tau$. We also know that $U^\mu(x_{n+1}) = U^{\mu_1}(x_{n+1}) > U^{\mu_1}(a)$. Hence for the linear function $\Phi(t) := U^{\mu_t}(x_{n+1}) - U^{\mu_t}(a)$ we have $\Phi(1) > 0$ and also $\Phi(\tau) \leq 0$ ($\tau > 1$). This yields $\Phi(0) > 0$, i.e., $U^{\mu'}(x_{n+1}) = U^{\mu_0}(x_{n+1}) > U^{\mu_0}(a) = U^{\mu'}(y)$ for all $y \in K$. We have therefore shown that all energy minimising (invariant) measures on K must defy the maximum principle.

Let now ν be an invariant measure on X . We have

$$\begin{aligned} M(X) &= U^\nu(y) = \sup_{x \in X} U^\nu(x) = D(X) \\ &\leq D(K) = \sup_{x \in K} U^{\mu'}(x) = U^{\mu'}(z) < U^{\mu'}(x_{n+1}) \end{aligned}$$

for all $y \in X$, $z \in K$. Thus we can conclude $U^\nu(y) \leq U^{\mu'}(y)$ for all $y \in X$ and even “ $<$ ” for $y = x_{n+1}$. Integrating with respect to ν would yield

$$\int_X \int_X k \, d\nu \, d\nu = M(X) < \int_X \int_X k \, d\mu' \, d\nu = \int_X \int_X k \, d\nu \, d\mu' = M(X),$$

hence a contradiction, unless $\nu(\{x_{n+1}\}) = 0$. If $\nu(\{x_{n+1}\}) = 0$ held, then ν would be an energy minimising measure on K . This is because obviously $\text{supp } \nu \subseteq K$ holds, and the potential of ν is constant $M(X)$ over K , so

$$M(X) = \int_K \int_K k \, d\nu \, d\mu' = \int_K \int_K k \, d\mu' \, d\nu = M(K) \quad \text{holds.}$$

As we saw above, then ν would not satisfy the maximum principle, a contradiction again, since the potential of ν is constant on X . The proof of the case of finite X is complete.

We turn now to the general case of X being a locally compact space with continuous kernel. Let μ be a compactly supported probability measure on X and $y \notin \text{supp } \mu$. Set $K = \text{supp } \mu$ and note that both $\mathfrak{M}_1(K) \ni \nu \mapsto \sup_K U^\nu$ and $\nu \mapsto U^\nu(y)$ are continuous mappings with respect to the weak*-topology on $\mathfrak{M}_1(K)$. If $\sup_K U^\mu < U^\mu(y)$ were true, we could therefore find, by a standard approximation argument, see for example [6, Lemma 3.8], a finitely supported probability measure μ' on K for which

$$\sup_{x \in \text{supp } \mu'} U^{\mu'}(x) \leq \sup_{x \in K} U^{\mu'}(x) < U^\mu(y).$$

This is nevertheless impossible by the first part of the proof, thus the assertion of the theorem follows. \blacksquare

Acknowledgement

The authors are deeply indebted to Szilárd Révész for his insightful suggestions and for the motivating discussions.

References

1. Anagnostopoulos, V. and Sz. Gy. Révész: 2006, ‘Polarization constants for products of linear functionals over \mathbb{R}^2 and \mathbb{C}^2 and Chebyshev constants of the unit sphere’. *Publ. Math. Debrecen* **68**(1–2), 75–83.

2. Bourbaki, N.: 1965, *Intégration, Éléments de Mathématique XIII.*, Vol. 1175 of *Actualités Sci. Ind.* Paris: Hermann, 2nd edition.
3. Carleson, L.: 1967, *Selected Problems on Exceptional Sets*, Vol. 13 of *Van Nostrand Mathematical Studies*. D. Van Nostrand Co., Inc.
4. Choquet, G.: 1958/59, ‘Diamètre transfini et comparaison de diverses capacités’. Technical report, Faculté des Sciences de Paris.
5. Farkas, B. and Sz. Gy. Révész: 2005, ‘Rendezvous numbers in normed spaces’. *Bull. Austr. Math. Soc.* **72**, 423–440.
6. Farkas, B. and Sz. Gy. Révész: 2006a, ‘Potential theoretic approach to rendezvous numbers’. *Monatshefte Math.* **148**, 309–331.
7. Farkas, B. and Sz. Gy. Révész: 2006b, ‘Rendezvous numbers of metric spaces – a potential theoretic approach’. *Arch. Math. (Basel)* **86**, 268–281.
8. Fekete, M.: 1923, ‘Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzahligen Koeffizienten’. *Math. Z.* **17**, 228–249.
9. Fuglede, B.: 1960, ‘On the theory of potentials in locally compact spaces’. *Acta Math.* **103**, 139–215.
10. Fuglede, B.: 1965, ‘Le théorème du minimax et la théorie fine du potentiel’. *Ann Inst. Fourier* **15**, 65–87.
11. Gross, O.: 1964, ‘The rendezvous value of a metric space’. *Ann. of Math. Stud.* **52**, 49–53.
12. Landkof, N. S.: 1972, *Foundations of modern potential theory*, Vol. 180 of *Die Grundlehren der mathematischen Wissenschaften*. New York, Heidelberg: Springer.
13. Mhaskar, H. N. and E. B. Saff: 1992, ‘Weighted analogues of capacity, transfinite diameter and Chebyshev constants’. *Constr. Approx.* **8**(1), 105–124.
14. Morris, S. A. and P. Nickolas: 1983, ‘On the average distance property of compact connected metric spaces’. *Arch. Math.* **40**, 459–463.
15. Ohtsuka, M.: 1961, ‘On potentials in locally compact spaces’. *J. Sci. Hiroshima Univ. ser A* **1**, 135–352.
16. Ohtsuka, M.: 1965, ‘An application of the minimax theorem to the theory of capacity’. *J. Sci. Hiroshima Univ. ser A* **29**, 217–221.
17. Ohtsuka, M.: 1967, ‘On various definitions of capacity and related notions’. *Nagoya Math. J.* **30**, 121–127.
18. Pólya, Gy. and G. Szegő: 1931, ‘Über den transfiniten Durchmesser (Kapazitätskonstante) von ebenen und räumlichen Punktmengen’. *J. Reine Angew. Math.* **165**, 4–49.
19. Révész, Sz. Gy. and Y. Sarantopoulos: 2004, ‘Plank problems, polarization, and Chebyshev constants’. *J. Korean Math. Soc.* **41**(1), 157–174.
20. Saff, E. B. and V. Totik: 1997, *Logarithmic potentials with external fields*, Vol. 316 of *Grundlehren der Mathematischen Wissenschaften*. Springer, Berlin.
21. Stadje, W.: 1981, ‘A property of compact, connected spaces’. *Arch. Math.* **36**, 275–280.
22. Thomassen, C.: 2000, ‘The rendezvous number of a symmetric matrix and a compact connected metric space’. *Amer. Math. Monthly* **107**(2), 163–166.
23. Wolf, R.: 1997, ‘On the average distance property and certain energy integrals’. *Ark. Mat.* **35**, 387–400.
24. Zaharjuta, V. P.: 1975, ‘Transfinite diameter, Chebyshev constants, and capacity for compacta in \mathbb{C}^n ’. *Math. USSR Sbornik* **25**(3), 350–364.